Lecture 1: What is QFT?

Content: Notation, Poincaré Groups, Unitary Transformations & Representations, Subgroups, Failure of Single-Particle RQM.

Notations and Conventions:

Work in units $c = \hbar = 1$

Thus $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$

Why QFT?

(i) Theory of everything (Quantum & gravity)

Quantum mechanics - refined off of probabilistic measurements, hence

$\langle H \rangle$ "Hamiltonian Expectation Value"

(ii) Relativistic Quantum Field Theory (Lower energies, larger scales) "Effective Field Theory"

Decoherence (noise)!

(iii) Non-Relativistic Quantum Field Theory (Even lower energies, larger scales)

→ Good for modelling condensed matter interactions (very cold temperatures!)

(iv) Classical Field Theory

Relativistic QFT: Groups

What does it mean to say a quantum theory is relativistic?

Ans: The theory is symmetric under the Poincaré group.

(Think about rotational invariances... ie, Lorentz Transformations)

Recall: if (x_0, x_1, x_2, x_3) are coordinates in an inertial reference frame. Then, in another reference frame, coordinates must satisfy the invariant interval

$$\eta_{\mu\nu}\, dx^\mu dx^\nu = \eta_{\mu\nu}\, dx'^\mu dx'^\nu \qquad (1.1)$$

$$\text{OR} \quad \eta_{\mu\nu} \frac{dx'^{\mu}}{dx^{\lambda}} \frac{dx'^{\nu}}{dx^{\kappa}} = \eta_{\lambda\kappa} \tag{1.2}$$

$$\text{where } \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

← Metric Tensor

& use summation convention

Any coordinate transformation satisfying (1.1) is linear (hence of the form)

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{1.3}$$

write as (Λ, a)

where a^{μ} is a constant 4-vector and $\Lambda^{\mu}{}_{\nu}$ is the Lorentz Transformation, and

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\kappa} = \eta_{\lambda\kappa} \tag{1.4}$$

$$\Longrightarrow \quad \Lambda^{T} \eta \Lambda = \eta .$$

These transformations form a group: $\mathcal{P}_4$

(i) "Identity" $\quad (\mathbb{I}, 0) \in \mathcal{P}_4 \quad$ (1.5)

with $(\mathbb{I}, 0) * (\Lambda, a) = (\Lambda, a)$

(ii) "Product" $\quad (\Lambda, a) * (\bar{\Lambda}, \bar{a}) \in \mathcal{P}_4 \quad$ (1.6)

(iii) "Inverse" $\quad (\Lambda', a') * (\Lambda, a) = (\mathbb{I}, 0)$ $\quad$ (1.7)

i.e. $\forall (\Lambda, a)$, $\exists (\Lambda', a')$ s.t. (1.7) holds

For instance, if we take out

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} + a^{\mu}$$

and then $\quad x''^{\mu} = \bar{\Lambda}^{\mu}_{\ \kappa} x'^{\kappa} + \bar{a}^{\mu}$

then the total effect is

$$(\bar{\Lambda}, \bar{a}) * (\Lambda, a) \equiv (\bar{\Lambda}\Lambda, \bar{\Lambda} a + \bar{a}). \qquad (1.8)$$

PROOF: Substitute x'^{μ} into $x''^{\mu} = \bar{\Lambda}^{\mu}_{\ \kappa} x'^{\kappa} + \bar{a}^{\mu}$:

$$\begin{aligned} x''^{\mu} &= \bar{\Lambda}^{\mu}_{\ \kappa} \left(\Lambda^{\kappa}_{\ \nu} x^{\nu} + a^{\kappa} \right) + \bar{a}^{\mu} \\ &= \bar{\Lambda}^{\mu}_{\ \kappa} \Lambda^{\kappa}_{\ \nu} x^{\nu} + \bar{\Lambda}^{\mu}_{\ \kappa} a^{\kappa} + \bar{a}^{\mu} \end{aligned}$$

Hence $\quad (\bar{\Lambda}\Lambda, \bar{\Lambda} a + \bar{a})$. $\quad \blacksquare$

Relativistic Quantum Theory: Unitary Operators

- In QM, all symmetries are represented by either unitary or antiunitary operators.

To build a relativistic quantum theory, we need a

(i) (Seperable) Hilbert Space $\mathcal{H}$

(ii) To every $(\Lambda, a) \in \mathcal{P}_4$, we can find a unitary transformation

$$\mathcal{U}(\Lambda, a) : \mathcal{H} \to \mathcal{H}$$

such that

$\phi \in \mathbb{R}$

Just a phase

$$1^*.\quad \mathcal{U}(\mathbb{I}, 0) = e^{i\phi}\, \mathbb{I}$$

$$2^*.\quad \mathcal{U}(\bar{\Lambda}, \bar{a})\,\mathcal{U}(\Lambda, a) = e^{i\phi\left((\bar{\Lambda}, \bar{a}), (\Lambda, a)\right)} \cdot \mathcal{U}(\bar{\Lambda}\Lambda, \bar{\Lambda} a + \bar{a})$$

Infinitely Many Transformations

Any family $\mathcal{U}(\Lambda, a)$ which obeys (1^*), (2^*) is called a Projective Unitary Representation of $\mathcal{P}_4$.

Transformations which obey (i) and (ii) are <u>very difficult</u> to find.

<u>Why is Rel. QM so Hard?</u>

Theorem: There are no nontrivial finite-dimensional unitary representations of $\mathcal{P}_4$.

only infinite dimensional !

Example: SO(3) [Rotations] unitary representations are

$$\mathcal{U} = e^{ix\hat{J}^* + iy\hat{J}^* + iz\hat{J}^*} \qquad (1.9)$$

with J^*'s being angular momentum.

(A rotation is just a 3×3 matrix which satisfies $O^T O = \mathbb{I}$)

Subgroups:

Consider the set

"Time translation subgroup" of $\mathcal{P}_4$

$$V_t = \{(\mathbb{I}, (t,0,0,0)) : t \in \mathbb{R}\} \in \mathcal{P}_4.$$

Set of all translations which move us forward in time

If $\mathcal{U}$ is a unitary representation of $\mathcal{P}_4$, then

$$V(t) \equiv \mathcal{U}(\mathbb{I}, (t,0,0,0)) \qquad (1.10)$$

is a one-parameter family of unitaries

$$V(s)V(t) = V(s+t)$$

which solves Schrödinger's Equation

$$\frac{dV(t)}{dt} = i\hat{H}\,V(t) \qquad (1.11)$$

for some Self-adjoint Hamiltonian $\hat{H}$.

We require that $\mathcal{U}(\Lambda, a)$ is "Positive energy"

ie. $\mathrm{spec}(\hat{H}) \subsetneq \mathbb{R}^+$

Spectrum of $\hat{H}$ // Eigenvalues of $\hat{H}$

Else things are unstable (particles continue to decay)

<u>Good news</u>: All single-particle unitary representations of $\mathcal{P}_4$ have been classified (by Wigner) and are labelled by two numbers:

1. mass
2. spin

} invariant under $\Lambda_\mu^{\ \nu}$.

Bad news: there is a tension between locality and interactions

(ie, Nature consists of more than 1 particle !!!)

New particles must be created...